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THE PROBLEM OF INTERFACE HEAT EXCHANGE BETWEEN A GAS BUBBLE AND A LIQUID†

N. A. ZOLOVKIN, A. G. PETROV and N. S. KHABEYEV

Moscow

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The problem of radial oscillations of a spherical gas bubble in an unbounded volume of an incompressible liquid under homobaric conditions in the gas [1, 2] is considered. The heat flux at the interface is investigated in the form of a Duhamel integral. The kernel of the integral is represented by a series of exponential functions, and a simple analytic approximation is obtained for it with high accuracy. The principal asymptotic expansions for the heat flux are obtained for high and low Peclet numbers. Expressions are derived for the remainder terms of the expansion.

There are a considerable number of theoretical and experimental publications devoted to an investigation of the effect of interphase heat exchange on the oscillations of gas bubbles in a liquid. A detailed discussion of these can be found in [1–3].

1. THE FUNDAMENTAL EXPRESSION FOR THE INTERFACE HEAT FLUX

When there are no phase transitions, the temperature of the liquid remains practically unchanged and the heat flux q at the interface is produced exclusively by the thermal resistance of the gas [4]. Because there are no phase transitions, the heat flux, naturally, is continuous. Hence, q is determined by solving the internal problem of heat exchange for a bubble. The following expression is obtained from the solution of the linearized heat-conduction equation in a gas by Fourier's method [3]

$$q(t) = -\frac{R_0}{\pi^2 t_*}\int_0^t \frac{dp}{dt_1} G\left(\frac{t-t_1}{t_*}\right) dt_1 \tag{1.1}$$

Here R_0 is the equilibrium radius of the bubble, t is the time, p is the pressure in the gas, $q = \lambda\, \partial T/\partial r$ is the interface heat flux, r is the Euler coordinate—the distance from the centre of the bubble, T and λ are the temperature and thermal conductivity of the gas, $t_* = R_0^2/(\pi^2 a)$ is the characteristic thermal time of the problem, and a is the thermal diffusivity of the gas.

The kernel of integral (1.1) can be written as follows:

$$G(x) = 2\sum_{n=1}^{\infty} \exp(-n^2 x) = \psi(x) - 1, \quad \psi(x) = \sum_{n=-\infty}^{\infty} \exp(-n^2 x) \tag{1.2}$$

2. ANALYTIC APPROXIMATION OF THE INTEGRAND

The function $\psi(x)$ can be expressed in terms of the theta-function, and the following identity holds for it [5]

$$\psi(x) = \sqrt{\pi/x}\, \psi(\pi^2/x) \tag{2.1}$$

From (2.1) we can obtain an expansion that is convenient for calculating $\psi(x)$ when $x < x_1$

$$\psi(x) = \sqrt{\pi/x} + r_1(x)$$
$$r_1(x) = 2\sqrt{\pi/x}\left(\exp(-\pi^2/x) + \exp(-4\pi^2/x) + \ldots\right) \tag{2.2}$$

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From (1.2) we can obtain an expansion which is convenient for calculating $\psi(x)$ when $x > x_1$

$$\psi(x) = 1 + 2\exp(-x) + r_2(x)$$
$$r_2(x) = 2(\exp(-4x) + \exp(-9x) + \ldots) \tag{2.3}$$

The point x_1 can be chosen so that the remainder terms of r_1 and r_2 are simultaneously the least. It is obvious that the point x_1 will satisfy the equation $r_1(x) = r_2(x)$, whence we obtain $x_1 = 1.526$. As a result we obtain the following analytic approximation for the function $G(x)$

$$G(x) = \begin{cases} \sqrt{\pi / x} - 1, & 0 < x \leq x_1 \\ 2\exp(-x), & x_1 \leq x \end{cases} \tag{2.4}$$

The highest error of approximation (2.4) occurs at the point x_1: $r_1(x_1) = r_2(x_1) = 0.0045$ ($G(x_1) = 0.44$), i.e. the relative error of approximation (2.4) amounts to 1%.

The second approximation for $G(x)$, which follows from expansions (2.2) and (2.3), has the form

$$G(x) = \begin{cases} \sqrt{\pi / x}\left(1 + 2\exp(-\pi^2 / x)\right) - 1, & 0 < x \leq x_2 \\ 2\exp(-x) + 2\exp(-4x), & x_2 \leq x \end{cases} \tag{2.5}$$

The boundary point x_2 is found in the same way from the condition for the error to be a minimum: $x_2 = 2.083$, where the greatest relative error amounts to 5.6×10^{-6}%.

The process of obtaining more accurate approximations can be continued similarly. The maximum error of the kth approximation decreases as $\exp(-(k+1)^2 x_k)$. It can be shown that the quantity x_k increases monotonically and, in the limit as $k \to \infty$, approaches π.

Approximation (2.4) was proposed previously in [6], in which $\pi/2$ was taken instead of the point x_1.

3. ASYMPTOTIC FORMS FOR HIGH AND LOW PECLET NUMBERS

Expression (1.1), taking (2.4) into account, can be written in the form

$$q(t) = -\frac{R_0}{\pi^2 t_*}\left(2\int_0^{t-x_1 t_*} \frac{dp}{dt_1}\exp\left(-\frac{t-t_1}{t_*}\right)dt_1 + \sqrt{\pi t_*}\int_{t-x_1 t_*}^{t} \frac{dp}{dt_1}\frac{dt_1}{\sqrt{t-t_1}} - \int_{t-x_1 t_*}^{t}\frac{dp}{dt_1}dt_1\right), \quad t > x_1 t_* \tag{3.1}$$

Here p_0 is the equilibrium pressure in the gas and in the liquid, ρ_1 is the density of the liquid (ρ_1 = const), $\omega_r = (3\gamma p_0/\rho_1)^{1/2}/R_0$ is the Minnaert frequency of free adiabatic oscillations of the gas bubble [2], and $\mathrm{Pe} = 2\pi^2\omega_r t_* = (2R_0/a)(3\gamma p_0/\rho_1)^{1/2}$ is the thermal Peclet number in the gas, which represents the square of the ratio of the characteristic dimensions of the problem to the thickness of the temperature boundary layer in the gas. If the oscillations of the bubble are close to isothermal, the Peclet number approaches zero and, conversely, for oscillations close to adiabatic we have $\mathrm{Pe} \gg 1$.

We will consider small sinusoidal oscillations of the bubbles. The pressure of the gas varies periodically in the form of the real part of the expression

$$p(t) = p_0(1 + \sigma\exp(i\omega_r t)), \quad |\sigma| \ll 1 \quad (\sigma = \Delta p / p_0)$$

Here Δp is the amplitude of the oscillations (for simplicity we will assume Δp = const).

After replacing the variables of integration $y = \omega_r(t - t_1)$, expression (3.1) takes the form ($\tau = t\omega_r$ is dimensionless time)

$$q(t) = -i\sigma\frac{R_0 p_0}{\pi^2 t_*}\exp(i\tau)(I_1 + I_2 + I_3), \quad \tau > \tau_*, \quad \tau_* = x_1\frac{\mathrm{Pe}}{2\pi^2}$$

$$I_1 = 2\int_{\tau_*}^{\tau}\exp\left(-y(i + \frac{2\pi^2}{\mathrm{Pe}})\right)dy, \quad I_2 = \left(\frac{\mathrm{Pe}}{2\pi}\right)^{1/2}\int_0^{\tau_*}\frac{\exp(-iy)}{\sqrt{y}}dy, \quad I_3 = -\int_0^{\tau_*}\exp(-iy)dy$$

Here I_1, I_2 and I_3 correspond to the three terms in the expression for the heat flux (3.1).

The asymptotic form for $\mathrm{Pe} \gg 1$, $\tau \gg \mathrm{Pe}/\pi^2$ is

$$I_1 = \exp(-x_1)\frac{\mathrm{Pe}}{\pi^2} - i\exp(-x_1)(1 + x_1)\frac{\mathrm{Pe}}{2\pi^4} + \ldots$$
$$I_2 = \frac{\mathrm{Pe}}{\pi}\left(\frac{x_1}{\pi}\right)^{\frac{1}{2}} - i\frac{2}{3}\left(\frac{x_1}{\pi}\right)^{\frac{3}{2}}\left(\frac{\mathrm{Pe}}{2\pi}\right)^2 + \ldots \tag{3.2}$$
$$I_3 = -\tau_* + i\tau_*^2 / 2 + \ldots$$

This asymptotic form above shows that, in the case of the oscillations of a bubble, close to isothermal ($\mathrm{Pe} \ll 1$), it is necessary to take into account in (3.1) all three terms in the braces.

The asymptotic form for $\mathrm{Pe} \gg 1$, $\tau \gg \mathrm{Pe}/\pi^2$ is

$$I_1 = -2\exp(-x_1)(\sin\tau_* + i\cos\tau_*) + \ldots$$

$$I_2 = \left(\frac{\mathrm{Pe}}{4}\right)^{\frac{1}{2}} + \left(\frac{\pi}{x_1}\right)^{\frac{1}{2}} \sin\tau_* - i\left(\left(\frac{\mathrm{Pe}}{4}\right)^{\frac{1}{2}} - \left(\frac{\pi}{x_1}\right)^{\frac{1}{2}} \cos\tau_*\right) + \ldots \tag{3.3}$$

$$I_3 = -\sin\tau_* + i(1 - \cos\tau_*)$$

It follows from (3.3) that the sum of the integrals $I = I_1 + I_2 + I_3$ has the following principal asymptotic form: $I = \sqrt{(\mathrm{Pe})}/2 + O(1)$.

Hence, for the oscillations of a bubble close to adiabatic with Minnaert frequency, we obtain from (3.1), using (3.3), the following expression for the interface heat flux (1.1)

$$q(t) = -\sqrt{a/\pi} \int\limits_{t - x_1 t_*}^{t} \frac{dp}{dt_1} \frac{dt_1}{\sqrt{t - t_1}} \left(1 + O\left(\mathrm{Pe}^{-\frac{1}{2}}\right)\right) \tag{3.4}$$

Expression (3.4), as can be seen from (3.3), has an error of the order of $\mathrm{Pe}^{-1/2}$. Hence, it holds when $\sqrt{(\mathrm{Pe})} \gg 1$.

The asymptotic expressions (3.2) and (3.3) hold when $\tau \gg \mathrm{Pe}/\pi^2$. If τ is taken into account, terms containing the exponentially small factor $\exp(-2\pi^2\tau/\mathrm{Pe})$ are added to the expansion for I_1.

It has been shown in [6] how, using the asymptotic form (3.4), one can obtain the well-known Chapman–Plesset solution (the expression for the log decrement of the damping radial oscillations of a gas bubble) [7], constructed earlier by a more complex method, without using expression (3.4).

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